Topology-Preserving Map Generalization

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Abstract

Our objective in this paper is to provide a mathematical framework in which to discuss automating the process of map generalization. To that end, we explore the mathematical notion of topological equivalence (homeomorphism) and the mathematical methods of continuous deformation (homotopy) in a new mathematical/cartographic context especially designed to model generalization. We define our map features to permit ourselves to characterize scale change transformations of map generalization in the language of continuous mathematics. We show how typical map generalization operations can be realized as smoothly deformed topological equivalence relations. We utilize a system of scale-dependent color changes that accomplish the standard feature removal operations of map generalization without changing the map's underlying point-set topology. We then focus on constructing homotopy operations to accomplish the seeming structural and point-positional changes needed to transform our map into a generalized smaller-scale version of itself.

1 Introduction

There are many different mathematical models of a map, and the models highlight different map properties. Some mathematical models treat a map as a cell complex, for example. These cell-complex models are equipped to handle tests of a map's topological consistency and correctness, on one hand, but, on the other hand, they cannot shed any light on the position and extent of the map's features. We will describe more extensive mathematical models of maps, of map features, and of map changes that all incorporate geometry of position and extent. Our model of map changes will preserve topology (and, hence, maintain topological consistency and correctness) throughout a continuum of scale change transformations.

We will first study one set of considerations for drawing a single map. We will define map features and consider how those features get placed or distributed on a map. After we explore the mathematics of feature placement on a single map, we will look at another set of considerations for drawing not only a single map, but also for producing *all* smaller-scale generalizations derived from that map.

1.1 Lines on the Earth versus Lines on a Map

A map is a model of the Earth. Does that make our mathematical model of a map also a mathematical model of the Earth? In one very important way, our mathematical model of a map is not a good model of the Earth itself. That limitation has to do with the thickness of lines. Some linear features on the Earth do not have a thickness associated with them and some linear features do. There are *mathematically linear* one-dimensional features (features representing real-world entities that truly have no width, such as national boundary lines) and there are *nominally linear* features (features representing "line-like" entities that actually have an on-the-ground width, such a roads). On a map, however, all drawn lines have non-zero thickness and all features, to be visible, must occupy non-zero area. While some mathematical models of a map ignore this reality, a mathematical model of a map used for map generalization should take feature area and line thickness explicitly into account, and our mathematical map model does.

2 Our Mathematical Model of a Map

The mathematical model that we will employ for a map differs from some traditional math models in the following essential way: in our model, all *represented* and visibly discernable map features, whether they are so-called point features, line features, or region features, have a 2-dimensional non-zero-area point set associated with them. Line features occupy area, and point features do too. All of our *line features* consist of two com-

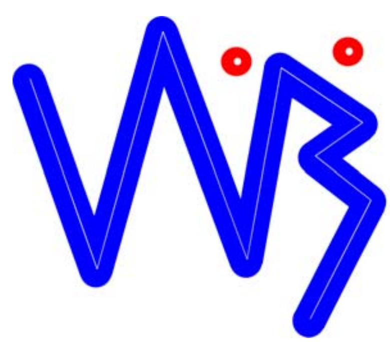

Figure 1: Line features are buffered polylines. Point features are disks.

ponents: (1) a 1-D underlying polyline and (2) an ϵ-buffer[1] around the polyline, resulting in a closed region surrounding the polyline that has uniform thickness 2ϵ and is rounded on the ends and joints (see figure 1). For technical reasons, we require the 1-D underlying polyline contain its

[1] An ϵ-buffer of a set S is the set of all points whose distance to some point in S is less than or equal to ϵ. An ϵ-buffer of any set is a union of closed disks of radius ϵ. Different map polylines may be assigned different values of ϵ (*i.e.*, different thicknesses).

endpoints, or equivalently that it be a closed[2] set in the plane. Similarly what we call a *point feature* will consist of two components as well: (1) a single point surrounded by (2) an ϵ buffer, resulting in a closed disk of diameter 2ϵ. In our mathematical map drawing model, we will refer to

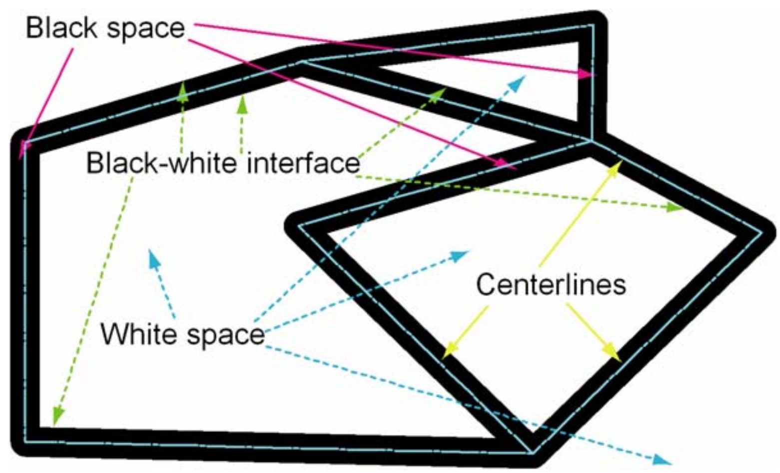

Figure 2: Black space, white space, the black-white boundary, and centerlines

the areas occupied by point and linear features as *drawn space*, or *black space*, and we will refer to the complement made up of regions delimited by the (thickened) point and line features as *white space*. To complete our mathematical map drawing model, we will introduce new truly 0-D and 1-D elements that have zero area, namely, the lines and points that make up the *black-white boundary*[3]. The lines and points that constitute the thickened line features' *centerlines*, as we have illustrated in Fig. 2, will continue to be part of our model, and, hence, will continue to be able to be referenced. The centerlines (and the point feature centers) are part of the black space, and they form a skeleton or medial axis of the black space. We will take care to formally define what can and cannot occur in our model, keeping in mind the application that we intend to model and its own formal rules.

2.1 Rules for Thickened-Feature Maps

To better understand the essential interplay between the zero-area and the non-zero-area components of linear features, we start by considering a data model that holds only the non-zero-area linear feature information (*i.e.*, the sequences of coordinate pairs of polyline vertices). To this model we naïvely apply a computer drawing system that drives a pen

[2]Insisting that the underlying 1-D polyline be closed will guarantee that the regions delimited by the 1-D polylines or by the buffered polylines will always be open sets.

[3]A point is in the black-white boundary if it is the limit point of both a sequence of black points and a sequence of white points. The black-white boundary is a closed set that is the finite union of (1-D) line segments and arcs of circles.

plotter with a fixed pen thickness to draw straight line segments connecting consecutive vertices of the input polylines. What we have described is a polyline drawing system that ignores any interplay between polyline vertex placement and line thickness. By examining what can go wrong with such a system, we may identify constraints that may be placed on the data model or on the drawing system to prevent certain types of drawing errors.

Figure 3: Buffered polylines and buffered points must not change topology.

In essence, we would like the drawn lines to behave like the zero-area-lines and to unambiguously represent the zero-area-lines. Usually they do. (It is this fact that allows us to "draw" lines in the first place without having to explain what they mean!) For example, we recover the zero-area line represented by a non-zero-area drawn line by visualizing the thick line shrinking at some uniform rate until only the 1-D skeleton is left. The formal mathematical process just described is what mathematicians call a *deformation retract* of the buffered polyline to its centerline. This cannot always be done smoothly. For example, in Fig. 3 the blue surrounding buffer cannot be shrunk continuously to the polyline whereas the less thick blue surrounding buffer in Fig. 1 may be continuously deformed to its centerline.

2.2 Drawn Space Observations

[Synopsis: Thickened lines have the same homotopy type as their centerlines. The complement of the centerlines and point centers is in the same homeomorphism class (and, hence, has the same connectivity) as the complement of the drawn space.] Another important behavior for the zero-area polylines is "no intersection or overlap of segments except at segment endpoints." This zero-area polyline behavior translates into "no intersection or overlap of thickened segments except at endpoint disks, which, when they intersect, will coincide or completely overlap."

3 A Model for Map Generalization

[Synopsis: The base map has well-defined drawn space/black space. The

versions at smaller scale may recolor features when the features drop below visible thickness. Before recoloring of features becomes necessary, the drawn space/white space areas have their positions adjusted to accommodate as many visible features as possible. When visible features can no longer be accommodated, some (pattern of) features will be recolored and disappear. Those no-longer-visible features may then be shrunk to arbitrarily small non-zero thickness, all the while being maintained as a well-defined point set that stays homeomorphic to the base map's drawn space.] Before defining a deformation process that corresponds to scale-reducing generalization, we must choose and keep fixed one largest-scale map drawing as our starting point. Suppose that our starting point map M_x has scale x. Our goal is to describe the evolution of a collection $\{M_y\}$ of generalizations of M_x of all scales $y < x$.

3.1 Math Notions

[Synopsis: Define homeomorphism and homotopy in a way that allows us to keep all scaled-down versions of the (generalized) map homeomorphic to the original base map by means of a continuous deformation through decreasing scales. Note that the 1-skeleton contracts in general, but spreads out locally to to keep adjacent features from coalescing until there is no room left for them all to be seen. After features drop below the thickness threshold of visible differentiation of features, some of the features are recolored to appear as one with the background, although technically the point set corresponding to the feature in question remains topologically equivalent to the original feature.] A function $f : X \to Y$ from topological space X to topological space Y is called *continuous* at a point $x_0 \in X$ if for every open neighborhood N of $f(x_0)$, there is an open neighborhood M of x_0 such that $f(M) \subseteq N$.

A function $f : X \to Y$ from topological space X to topological space Y is called a *homeomorphism* if f is continuous, bijective (one-to-one and onto), and it has a continuous inverse function f^{-1}.

Two continuous functions $f_0 : D \to R$ and $f_1 : D \to R$ that have the same domain (D) and the same range (R) are said to be *homotopic* if there exists a continuous function $F : [0,1] \times D \to R$ such that $F((0,d)) = f_0(d)$ and $F((1,d)) = f_1(d)$ for all $d \in D$. The function F is called a *homotopy* from f_0 to f_1. F is also called a homotopy between f_0 and f_1, because *being homotopic* is an equivalence relation among the continuous functions from D to R, and the set of all such continuous functions that are homotopic to a function g are said to form the homotopy equivalence class $[g]$ of the function g.

One may regard the image of $f_0(D)$ as being transformed continuously over time into the image of $f_1(D)$ by regarding t as a time parameter and considering the sets $F(\{t\} \times D)$ for all values of $t \in [0,1]$. For any fixed value of t, the function $f_t : D \to R$ that sends d onto $F((t,d))$ is called the homotopy slice at t. It is easy to show that every such f_t is also homotopic to both f_0 and f_1.

If we fix a point $d \in D$ and let t vary from 0 to 1, then we will obtain the trajectory that starts at $f_0(d)$ at time $t = 0$ and finishes at the point $f_1(d)$ when $t = 1$.

We will be especially interested in the case in which $R = D$ and for which $f_0(d) = d$ for all $d \in D$, that is, f_0 is the identity map on D. Then, intuitively, the homotopy equivalence class of the identity map on D will consist of "everything that D may be continuously transformed into".

Now we have all of the machinery to define the term *deformation retract* mentioned in Section 2.1. A *deformation retract* from the space D to a subset $U \subseteq D$ is a homotopy $F : [0,1] \times D \to D$ from the identity map 1_D on D ($\forall d \in D, F((0,d)) = 1_D(d) = d$) to a function whose image lies in U ($\forall d \in D, F((1,d)) \in U$) such that each slice restricted to U gives the identity function on U ($\forall t \in [0,1], \forall u \in U, F((t,u)) = 1_U(u) = u$).

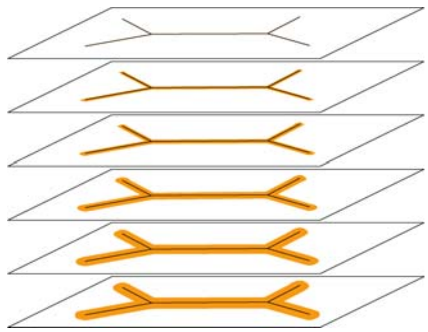

Figure 4: Thick polyline features continuously deformed to their 1-D centerlines.

4 Thick versus Thin

We have seen computer models that store line skeletons, but draw lines of non-zero thickness. We have also noted that lines drawn without conflict at one thickness may interfere with one another if drawn at a greater thickness. Why then do some computer models only check topology for infinitessimally thin lines? The answer may lie in the following statement of a theorem which we will prove after stating and proving a few lemmas.

Theorem 4.1 *Every finite cell decomposition of a rectangular region into 0-cells, 1-cells, and 2-cells for which the 0-cells are points and the 1-cells are polylines has an ϵ-buffer realization for some ϵ.*

5 Transformations of Thick-Featured Maps

5.1 Map-Shrinking versus Line-Thickening

Synopsis: Describe how map-frame-shrinking while keeping the same line thickness is equivalent to line thickening while keeping the same-size map frame. Discuss when the simple map-shrinking or line-thickening process breaks down–i.e., define critical or singular states of transition.

5.2 Critical States of a Scale Reduction

Synopsis: Define critical and not-critical states of transition: critical states occur when any thickening of black space without moving centerlines or changing color violates some minimum threshold condition. Define a collection of deformations of simple feature types that model how those features behave during non-critical and critical transitions.

5.3 Transformations During Non-critical States

Synopsis: Although line-thickening without any centerline movement is legitimate in this state, centerline movement might forestall reaching a critical state too soon.

5.4 Transformations During Critical States

6 Map Generalization Tools

Synopsis: In this section we describe how the generalized map space changes continuously with decreasing scale factor. We start out by handling totally smooth changes at non-singular transition points.