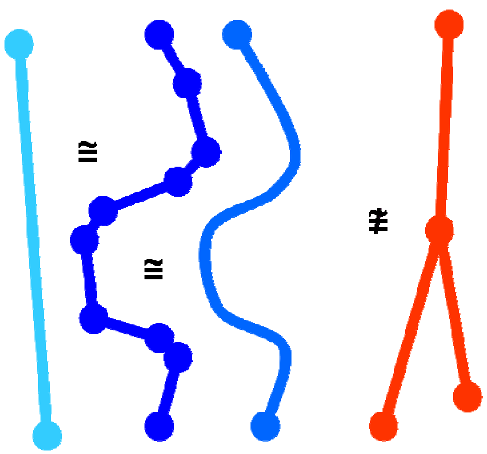
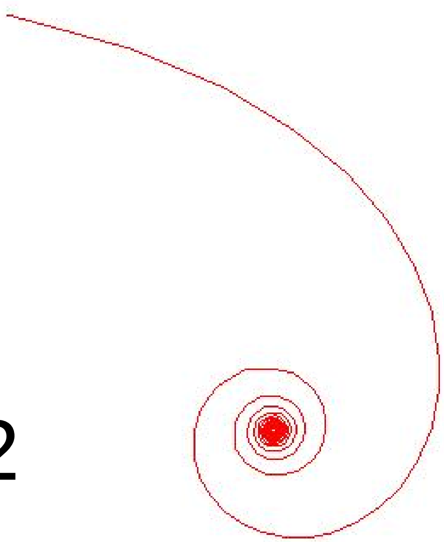
Mathematical Principles and Algorithms for Spatial Data



Alan J. Saalfeld

Tamal K. Dey

NMA201-01-1-2012



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THE OHIO STATE UNIVERSITY

Overview of today’s talk

- Contributions and focus of team members
- Some issues yet to be resolved
- Detailed algorithmic solution to a key topology problem (with supporting math)
- Discussion of how availability of science and math affects the choice of a model
- Possible alternative for selecting a particular mathematical model

Major Themes

- Piecewise linear approximation for curves and surfaces—when does it work well? When does it not work at all?
- What gets approximated? (The piecewise linear functions are dense in what larger set under what metric? What is the closure of the set of PLH maps?)
- How should we model a curve or a surface?

Tamal Dey's Research

- Meshing of (smooth) surfaces
  - Piecewise linear approximation
  - Topologically equivalent mesh surfaces
  - Meshes provide discretized versions of curvature, critical points (Morse theory)
  - Meshing is a natural construction to try on sampled data sets
  - Surface reconstruction from noisy point cloud data

Work with Satyan Devadoss

- Summer Program of Math Research for Undergraduates and Williams College
- Questioned assumptions and definitions of computational geometers (paper folding)
- “Paper has thickness” assumption in paper-folding corresponds to “Lines have thickness” assumption in map drawing.
- Map generalization under line thickening assumptions—homotopy methods.

Mathematics and Statistics

The SMALL Program

- *SMALL is a NSF-funded summer undergraduate math research program (REU) which is the largest and most successful in the country.*
- *About 15 – 25 undergraduates throughout the world spend 9 weeks on original research.*
- *Since its inception in 1988, over 60 students have published research articles.*

SMALL site at Williams

SMALL group 2001

SMALL 2004 Computational Cartography

- *Research Team:*

Jeff Danciger	*(UCSB, Creative Studies)*
John Mugno	*(Williams)*
Don Sheehy	*(Princeton)*
Rachel Ward	*(U.Texas)*

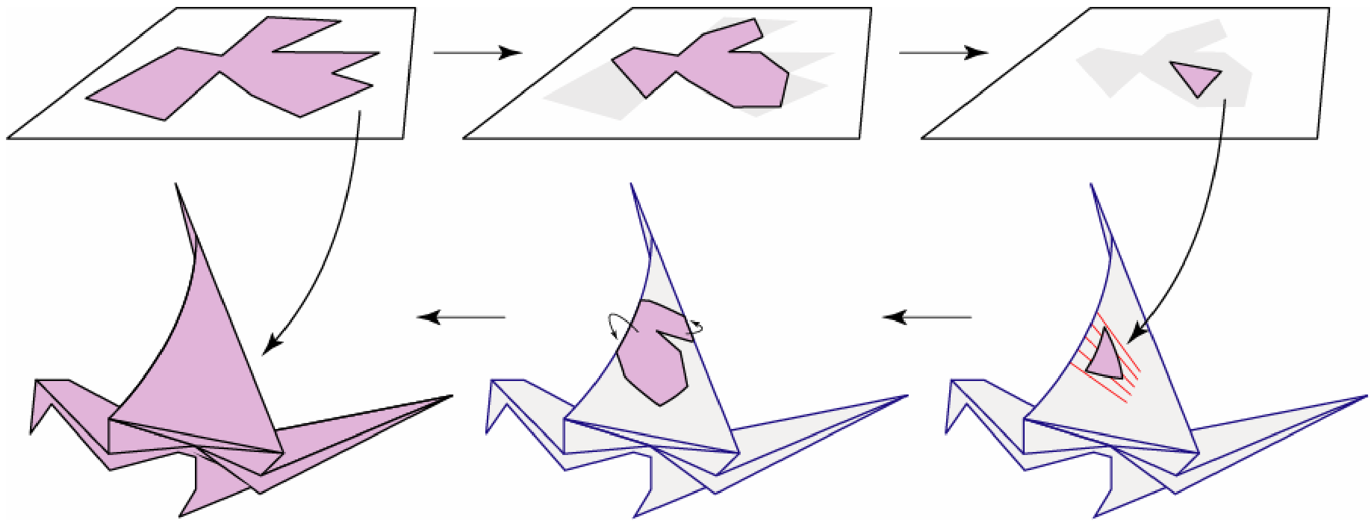
- *Collaboration with Ohio State team:*
  1. *Accountability of ideas.*
  2. *Tamal Dey visits in July.*
  3. *Sabbatical 2005-2006 at Ohio State*

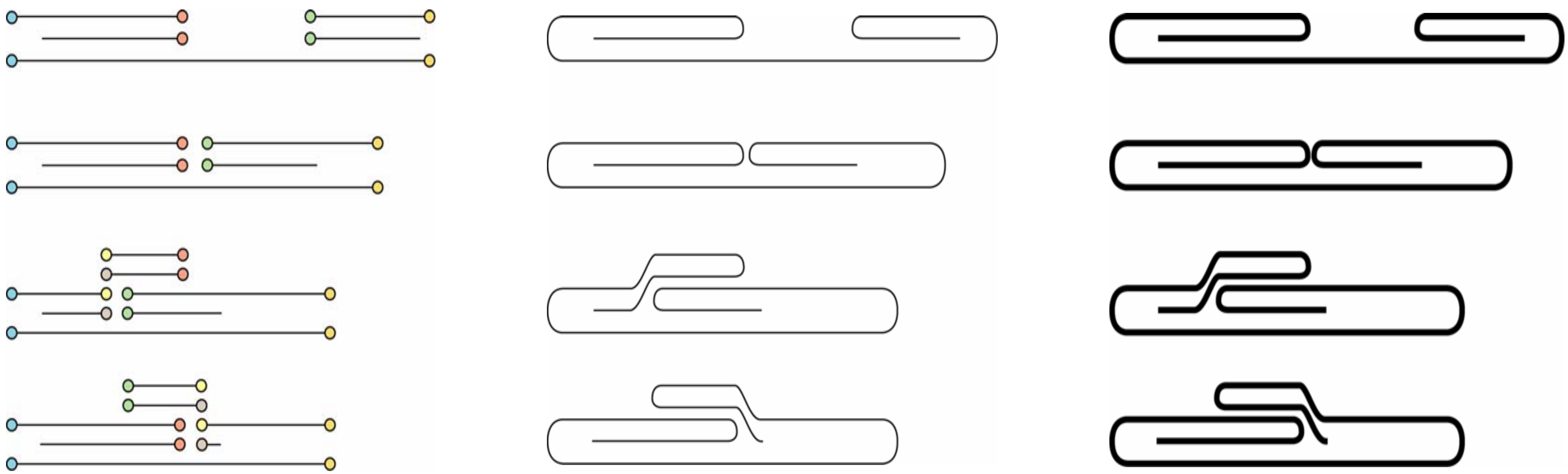
- *Thesis student 2004-2005*

- *Focus on three key problems in "Cartography":*
  1. *Averaging Point Clouds*
  2. *White Space Management*
  3. *Thick Origami*

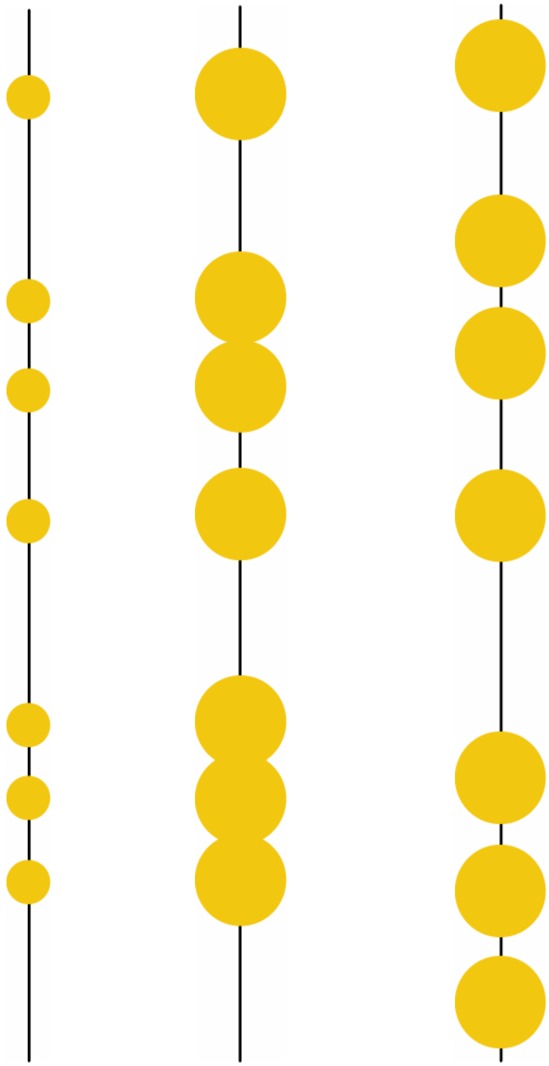
Thick Origami

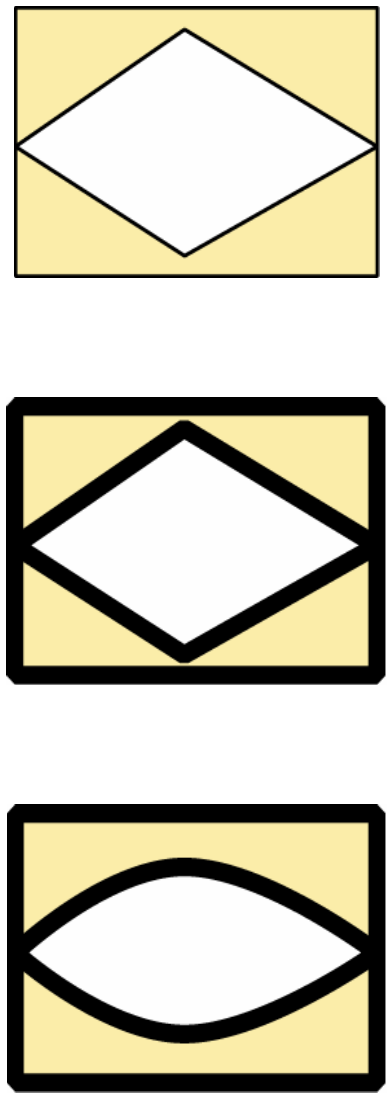
E. Demaine
S. Devadoss
J. Mitchell
J. O'Rourke

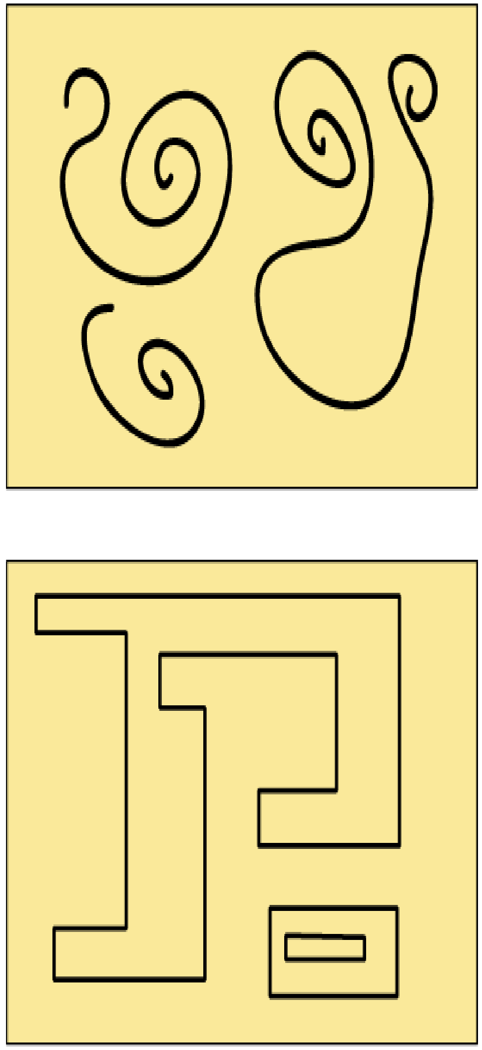




White Space





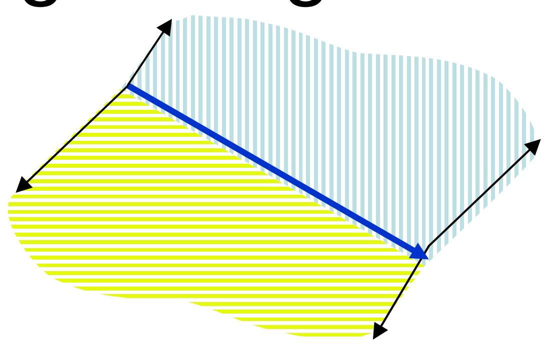


The Topology Problem(s)

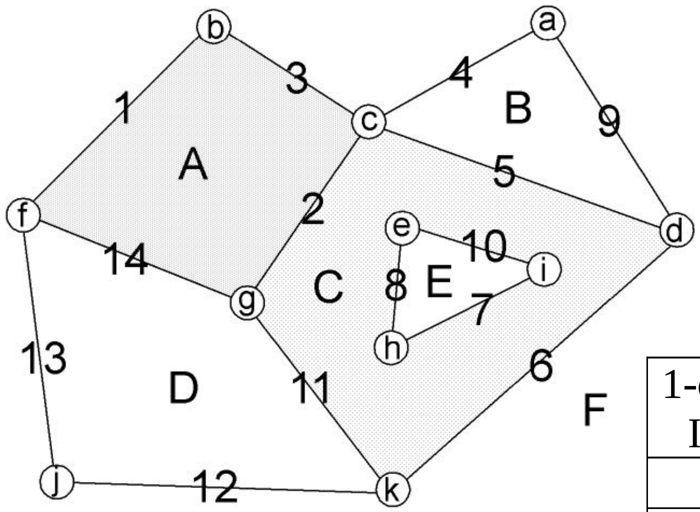
1. Mathematically define elementary **1D** map features (**0D** and **2D** features are easier)
2. Describe a **data structure** for encoding map features and their topological properties
3. Produce a **minimal** collection of algorithms for testing if an encoded set of map features is both **consistent** and **complete**
4. **Prove** that the algorithms constitute a **minimal** set of algorithms to check **completeness** & **consistency**

A Topology Solution

1. **1D** map features are polylines (0D are points and 2D are connected components of the complement)
2. A **data structure** for encoding map features and their topology: winged-edge data structure



The US Census Bureau's Topologically-Integrated Geographic Encoding & Referencing System (Winged-Edge Data Structure)



TIGER 1-cell file

					Next **1-cell** with the same:			
1-cell ID	From 0-cell	To 0-cell	Left 2-cell	Right 2-cell	From Right	To Left	From Left	To Right
1	b	f	A	F	3	14	3	13
2	c	g	C	A	3	11	5	14
3	b	c	F	A	1	4	1	2
4	a	c	B	F	9	5	9	3
5	c	d	B	C	2	9	4	6
6	d	k	F	C	5	12	9	11
7	h	i	E	C	8	10	8	10
8	e	h	E	C	10	7	10	7
9	a	d	F	B	4	6	4	5
10	e	i	C	E	8	7	8	7
11	g	k	C	D	14	6	2	12
12	j	k	D	F	13	11	13	6
13	f	j	D	F	1	12	14	12
14	f	g	A	D	13	2	1	11

TIGER 0-cell file

0-cell ID	Start 1-cell
a	4
b	1
c	2
d	5
e	8
f	1
g	2
h	7
i	7
j	12
k	6

TIGER 2-cell file

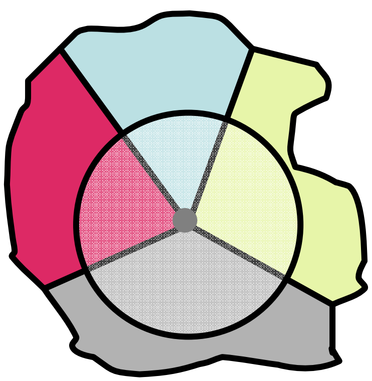
2-cell ID	Start 1-cell
A	1
B	4
C	2
D	11
E	7

A Topology Solution

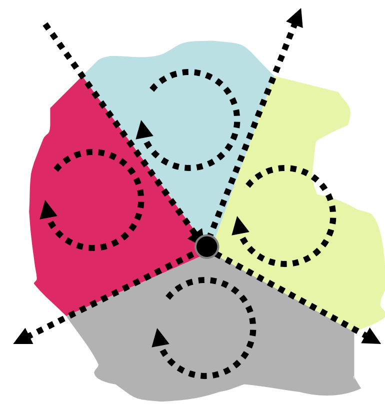
1. **1D** map features are polylines (0D are points and 2D are connected components of the complement)
2. A **data structure** for encoding map features and their topology: winged-edge data structure
3. A **minimal** collection of algorithms for testing a set of map features: umbrella edit, compute graph's connected components, Euler characteristic, test for orientability
4. **Proofs** that algorithms **work** and are **minimal:** local and global characterizations of surfaces by disk neighborhoods and a set of surface invariants, the Jordan Curve Theorem for [simple graph cycle]-to-[2-cell] correspondence and acyclic element characterizations

Computational topology of surfaces

- ***Correctness testing*** versus ***topology building***
  - Necessary and sufficient surface invariants
    - Vertex neighborhoods, Euler Characteristic, Orientability



$$\chi = \boldsymbol{f} - \boldsymbol{e} + \boldsymbol{v}$$



Doing things "Math-Backwards"

1. Define "primitives" based on their ease of mathematical representation and analysis: e.g., straight line segments, polylines.
2. Prove results for the elementary primitives
3. Check to see if proofs hold for, or extend to include, a larger class of primitive objects
4. Try to show that primitives are dense in a larger class of objects and adequately approximate objects in the larger class

Topology Theorems

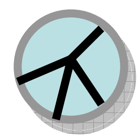
- Surface 1D/2D decomposition theorems
  - Jordan Curve Theorem
  - Schoenflies Curve Extension Theorem
  - Hauptvermutung (Common Refinement)
- Surface classification theorems
  - Local characterization of surface: disk-like
  - Invariance Theorems (Global Properties)
    - Euler Characteristic
    - Torsion (orientability)

Algorithms for Topology Editing

- Umbrella edit
  - Uses the winged-edge data structure
  - Builds a disk-like neighborhood at each 0-cell
- Euler characteristic edit
  - A union-find subroutine is used to compute the 1-skeleton's connected components
- Orientability edit
  - Each interior edge v_1v_2 has its vertices v_1 and v_2 enumerated v_1 before v_2 in one adjacent 2-cell and v_2 before v_1 in the other adjacent 2-cell

Umbrella edit

For every 0-cell **v** in the interior of the map:

Check that **v** is surrounded by a disk-like neighborhood  of a cycle of alternating 1-cells and 2-cells.

Euler characteristic edit

Count the 0-cells, 1-cells, and 2-cells: $\mathbf{n_0}$, $\mathbf{n_1}$, and $\mathbf{n_2}$, respectively.

Compute the number $\mathbf{n_{cc}}$ of connected components of the 1-skeleton. Let $\mathbf{n_{-1}=n_{cc}-1}$.

Verify that $\mathbf{n_2-n_1+n_0-n_{-1}= 2}$.

Connected component generation

Initialize every 0-cell to be its own component.

For each edge **vw:**

Merge the component of **v** with the component of **w.**

Merge-Find (Equivalence-Class Building from Equivalence Relations)

Initialize: For each point **v**, set parent of **v** equal to ϕ, $p(\mathbf{v}) := \phi$.

For some $i \geq 0$, $p^{i}(\mathbf{v}) \neq \phi$ & $p^{i+1}(\mathbf{v}) = \phi$.

We define $r(\mathbf{v})$ to be such a $p^{i}(\mathbf{v})$.

For each relation **vRw**:

If $r(\mathbf{v})=r(\mathbf{w})$ do nothing,

Else If $r(\mathbf{v})<r(\mathbf{w})$, $p(r(\mathbf{w})) := r(\mathbf{v})$,

Else $p(r(\mathbf{v})) := r(\mathbf{w})$.

Orientability edit

- Assign a rotation direction (e.g., ccw: ↺) to any 2-cell by cyclically ordering its 0-cells.
- Propagate the rotation direction across all 1-cells: ↺↑┃↓↺
- Check that all of the rotation directions propagated to any 2-cell across the 2-cell’s border are the same rotation direction.

Definitions: Primitives, Operations, Relations, **& Axioms** [of a Mathematical Model of Maps]

↓

Theorems [Verifiable Properties]

↓

Algorithms [Validation Methods]

↓

Computer Programs [Implementation]

} Recent Research Focus (Definitions & Axioms)

} Primary Research Focus (Theorems, Algorithms)

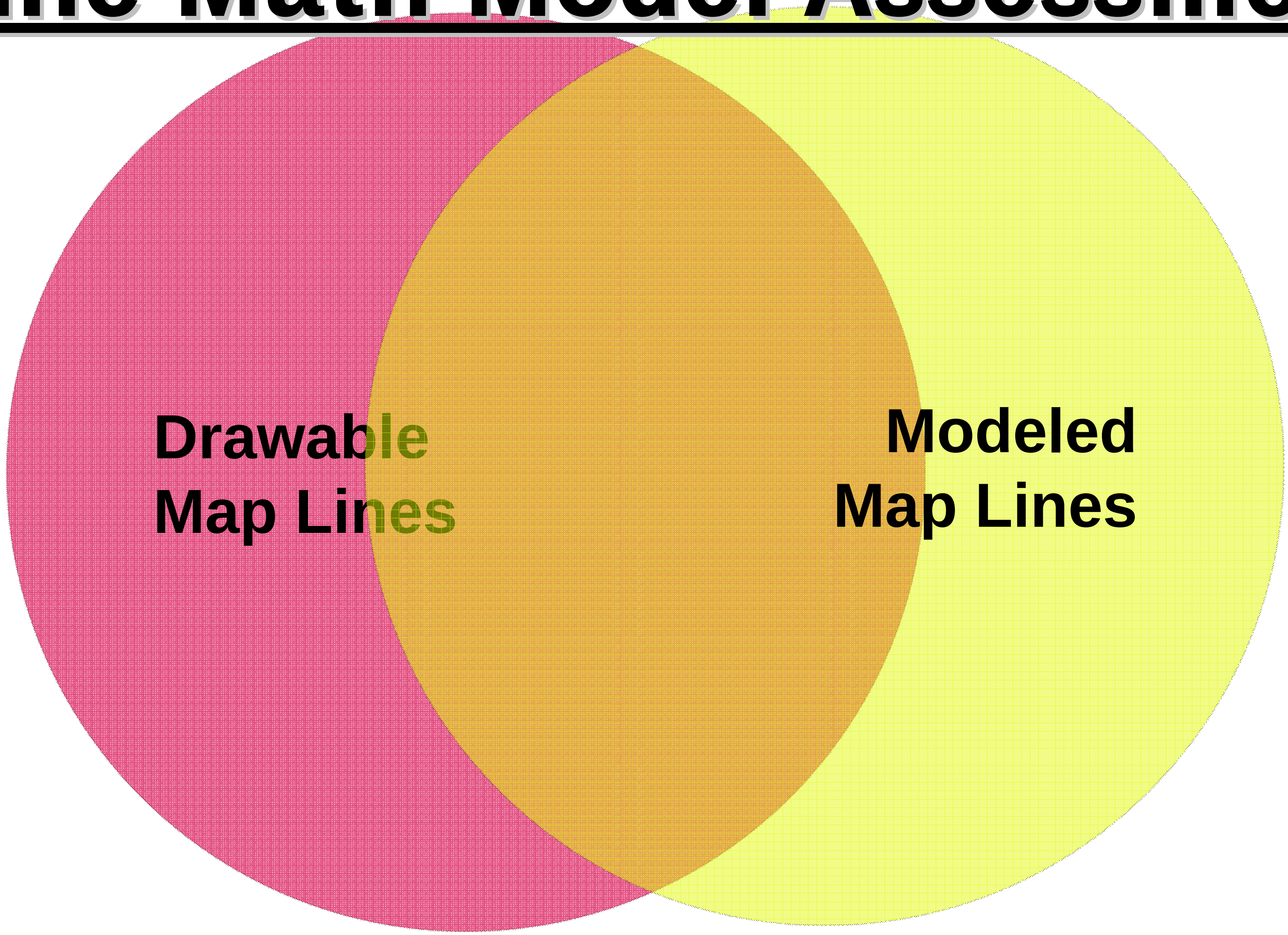
} Ongoing Development Focus (Computer Programs)

Map primitives

What math model to choose?

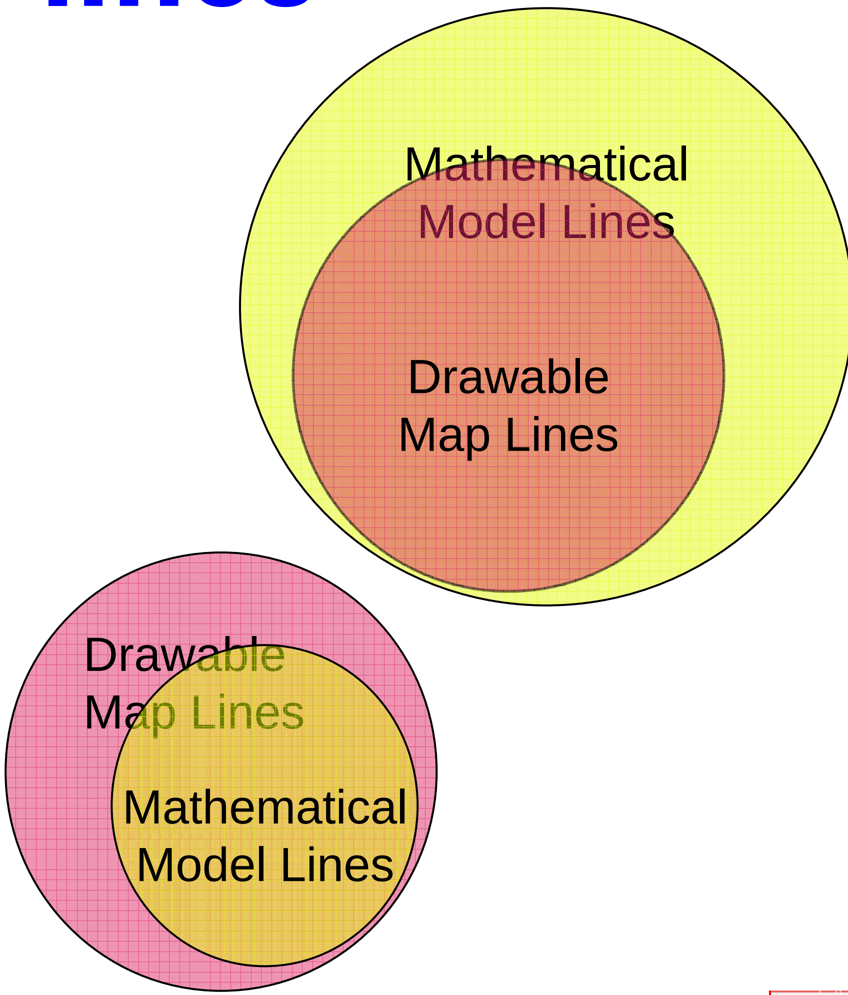
- Math-model primitives vs map primitives
  - How many & what kind of building blocks are needed?
- Covering all possible map drawing instances
  - E.g., Might contour lines be allowed to intersect?
- Identifying non-map-like math model behaviors
  - E.g., Lines might have no thickness, lines may have infinitely many intersections, infinite resolution, etc.
- Thick-line models & map generalization
  - White space is also “map real estate” to be managed.
- Proving model/map compatibility: How?

Line Math Model Assessment



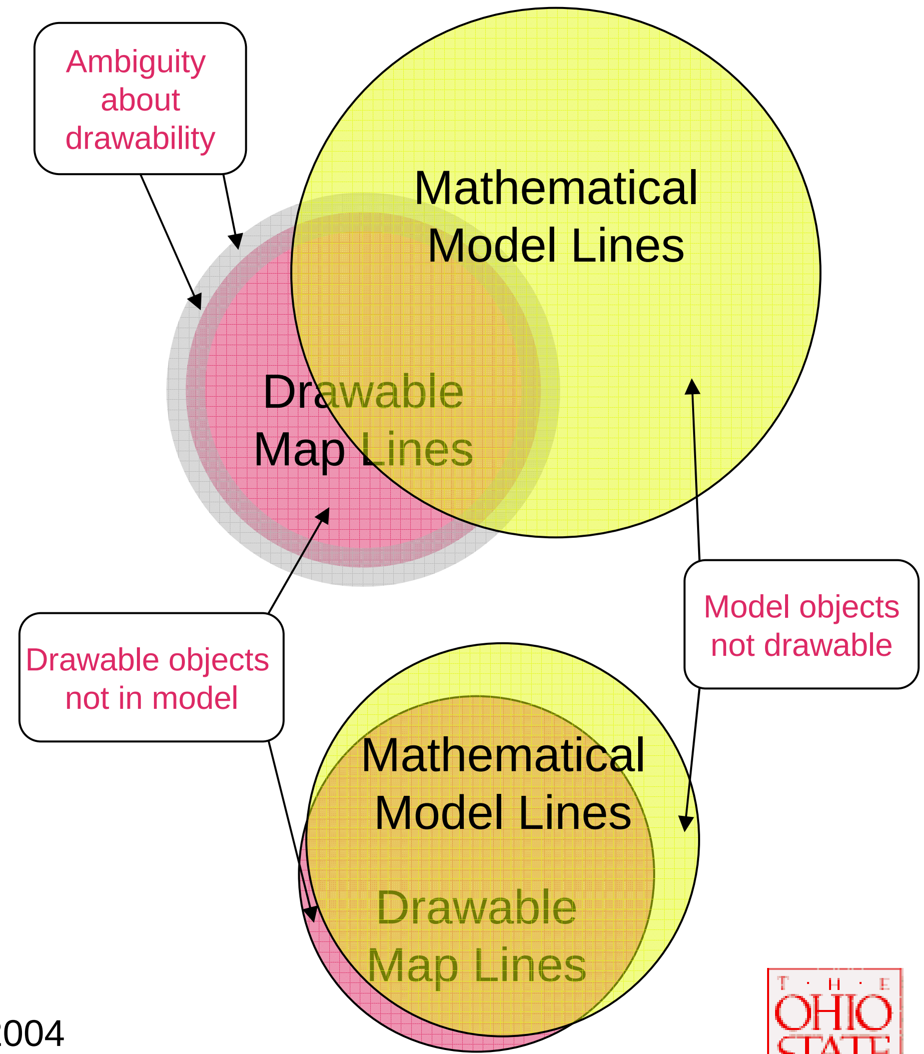
Math-model lines vs map lines

- Some math line models may be too broad
  - May encompass things other than map lines (e.g., continuous curves)
- Some math line models may be too narrow
  - May fail to account for certain types of map lines (e.g., polylines)



How do map lines behave?

- "Map drawing" itself is not analyzed scientifically
  - Exactly what we mean by "drawable" is not obvious nor is it easily defined
  - Mathematically formulated constraints on "drawability" can help reduce ambiguity
  - Mathematical properties of drawn lines can place limits on the math model



Identifying non-map-like behaviors of math models

- To try to list everything that could go wrong is inefficient and unscientific
  - No way of knowing when to stop
- Curve intersections may have only finitely many connected components
  - $\Rightarrow$ No infinite oscillations
- We'd like approximability by polylines
  - But what is actually being approximated?

Covering all possible drawing instances

- One might start with a physics model
  - How can a pen point move on or above a sheet of paper?
  - What is its track or trace?
  - What physical laws govern its motion?
    - Forces applied must change continuously or possibly even smoothly
    - Bounded length, Bounded speed, Bounded acceleration
    - Continuous motion

Proving model/map compatibility

- What does this mean mathematically?
- It could be taken to mean that the math from the physical system is consistent and compatible with the math from the model.

The End (of the talk)

- But not the end of this research…
- Any questions?
- Any thoughts on the subject of choosing good 1D primitive map objects? Let me know now or later.