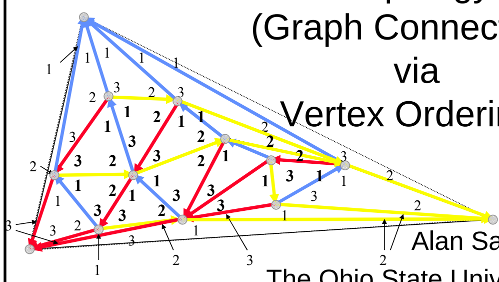

Topology (Graph Connectivity) via Vertex Orderings

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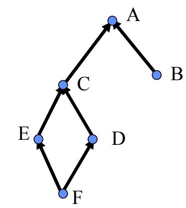
A Poset (partially ordered set) is a set S with a binary relation, denoted <, or ≺, that is:

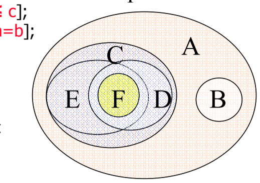
Transitive: [a < b and b < c implies a < c];
Antisymmetric: [a < b implies b is not < a];
and Irreflexive: [a is not < a].

Sometimes ≤ replaces **<** with the necessary changes in the properties. The relation ≤ is:

Transitive: [a **≤** b and b **≤** c implies a **≤** c];
Antisymmetric: [a **≤** b & b **≤** a implies a=b];
and Reflexive: [a **≤** a].

Set inclusion "⊂" or "⊆" provides the prototypical example of a poset. Every partial order ≺ is isomorphic to some set containment order ⊂.





We use the convention of arrow pointing to larger elements and when possible the arrows point up.

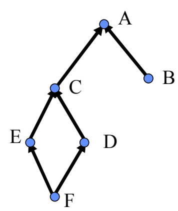
A subset I of a poset P that is closed under taking of lesser elements is called an ***ideal***: **a ∈ I and b ≤ a implies b ∈ I.**

A principal ideal I(p) of a poset element p is the set: **{x in P| x ≤ p}.**

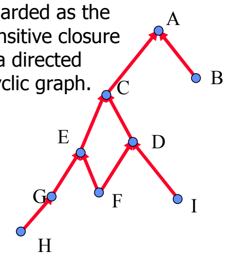
Every partial order **≤** is isomorphic to some set containment order ⊆ under **p ↦ I(p).**

A subset F of a poset is a set S that is closed under taking of larger elements is called a ***filter***: **a ∈ F and b > a implies b ∈ F.**

A principal filter F(p) of a poset element p is the set: **{x in P| p ≤ x}.**

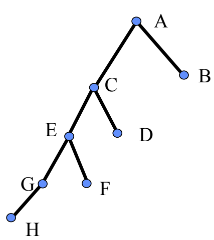


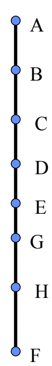
A poset can be regarded as the transitive closure of a directed acyclic graph.



≥	A	B	C	D	E	F	G	H	I
A	1	1	1	1	1	1	1	1	1
B	0	1	0	0	0	0	0	0	0
C	0	0	1	1	1	1	1	1	1
D	0	0	0	1	0	1	0	0	1
E	0	0	0	0	1	1	1	1	0
F	0	0	0	0	0	1	0	0	0
G	0	0	0	0	0	0	1	1	0
H	0	0	0	0	0	0	0	1	0
I	0	0	0	0	0	0	0	0	1

Covering relations or poset relations can always be represented by an upper triangular matrix of 0's and 1's showing ≥. The transitive closure of the covering relations matrix **M** is obtained by raising (**M**+**I**) to some power $(\mathbf{M}+\mathbf{I})^k$ (where *I* is the identity matrix, and addition and multiplication of entries are carried out in Boolean arithmetic).

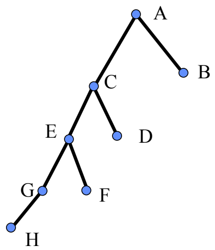




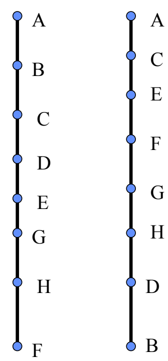
Observation: Every poset can be embedded in many different total or linear orders, each of which is sometimes called a topological sort. Every poset equals the intersection of all total orders that contain it. The dimension of a poset is the smallest number of total orders, the intersection of which is exactly the poset.

A few observations about dimension of posets:

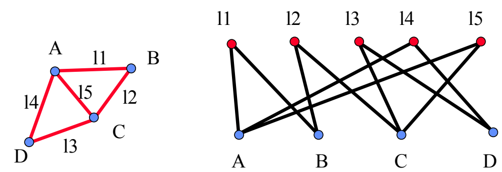
- A total order has dimension 1.
- A totally unordered set has dimension 2.
- A 4-colorable graph has dimension ≤ 4.
- A rooted tree has dimension 2.
- Determining if a poset has dimension 3 or more is NP-complete (Yannakakis--1982).
- Even posets of height 1 are complex.



Observation:
Every rooted tree has
(poset) dimension 2.

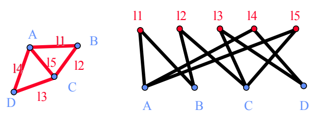


Inclusion order of a graph G(V,E) is a poset of height 1 (the only inclusions are: each edge contains its two end points):



Graph

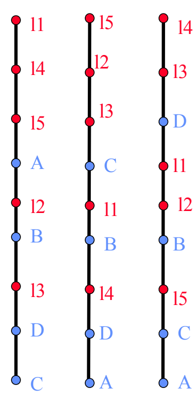
Poset of the graph

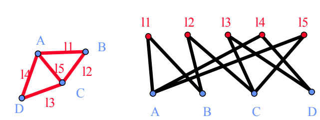


Graph

Poset of the graph

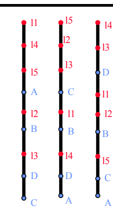
We can analyze the order dimension of a graph by examining total orders on the edge and vertex inclusion poset:



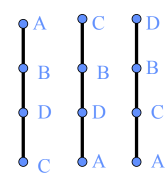


Graph

Poset of the graph



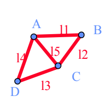
We can analyze the order dimension of a graph by examining total orders on the vertices alone provided those total orders have two properties called vertex and edge properties:
Vertex property: intersection of orders is empty.
Edge property: if (x,y) is an edge and z any other vertex, then there is some order for which both x<z and y<z.



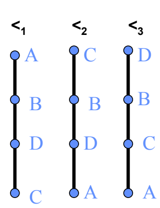
Edge property: if (x,y) is an edge and z any other vertex, then there is some order for which both x<z and y<z.

In terms of filters $\mathbf{F_i}$, the edge property says that if (x,y) is an edge, then we must have that all of the poset elements appear in the union:

$\bigcup_{i=1,\dots,d} [F_i(x) \cap (F_i(y)] = P$



Graph



Walter Schnyder's main result:

A graph is planar if and only if its edge-vertex inclusion poset has dimension ≤ 3.

The value that I envision for future research is not in the result itself, but in the tools that Schnyder developed to prove his result.

(⇐) How to draw a graph that realizes the 3 total orders (then do it compactly)
(⇒) How to convert a plane graph into 3 total orders on the vertices

(⇐) How to draw a graph in the plane that realizes the 3 total orders (later do it compactly)

Three total orders in "standardized" format have 3 maxima that are the two smallest values of the other two orders:

$<_1$: <u>B</u> <u>C</u> K J L F G V I E H D <u>A</u>
$<_2$: <u>A</u> <u>C</u> H D J I E K V F G L <u>B</u>
$<_3$: <u>A</u> <u>B</u> G L E D V F I H K J <u>C</u>

Three total orders
$<_1$: B C K J L F G V I E H D A
$<_2$: A C H D J I E K V F G L B
$<_3$: A B G L E D V F I H K J C

and their rank-order coordinates:

A = (1, 13, 13)
B = (13, 1, 12)
C = (12, 12, 1)
D = (2, 10, 8)
E = (4, 7, 9)
F = (8, 4, 6)
G = (7, 3, 11)
H = (3, 11, 4)
I = (5, 8, 5)
J = (10, 9, 2)
K = (11, 6, 3)
L = (9, 2, 10)
V = (6, 5, 7)

Three total orders
$<_1$: B C K J L F G V I E H D A
$<_2$: A C H D J I E K V F G L B
$<_3$: A B G L E D V F I H K J C

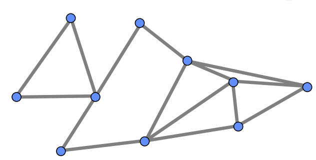
and their $2^{\text{rank-order}}$ $<_1$- and $<_2$-coordinates:

Lemma (Schnyder): If a line segment is drawn with these coordinates for every pair of vertices that satisfies the edge property, then no line segments will intersect except at a vertex of both.

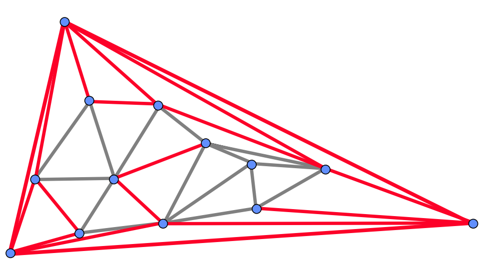
A = $(2^{1}, {}^{13})$
B = $(2^{13}, 2^{1})$
C = $(2^{12}, 2^{12})$
D = $(2^{2}, 2^{10})$
E = $(2^{4}, 2^{7})$
F = $(2^{8}, 2^{4})$
G = $(2^{7}, 2^{3})$
H = $(2^{3}, 2^{11})$
I = $(2^{5}, 2^{8})$
J = $(2^{10}, 2^{9})$
K = $(2^{11}, 2^{6})$
L = $(2^{9}, 2^{2})$
V = $(2^{6}, 2^{5})$

(⇒) How to convert a plane graph into 3 total orders on the vertices

Start with a Plane Graph



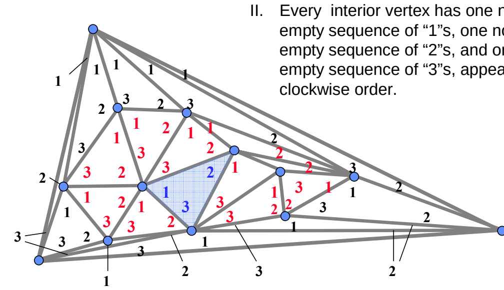
Embed the plane graph into a Triangular graph (Outer-planar triangulation)



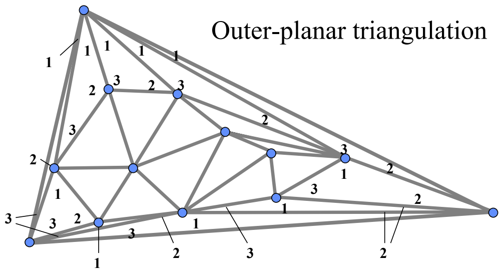
Normal Angle Labeling

(Existence proof by Schnyder)

I. Every triangle has all 3 labels appearing in clockwise order.
II. Every interior vertex has one non-empty sequence of "1"s, one non-empty sequence of "2"s, and one non-empty sequence of "3"s, appearing in clockwise order.

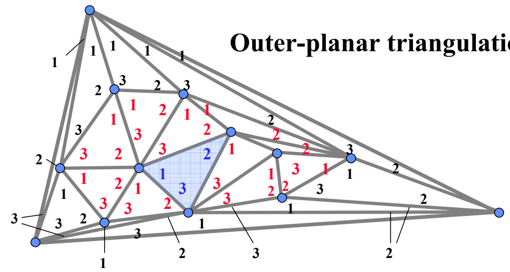


Outer-planar triangulation



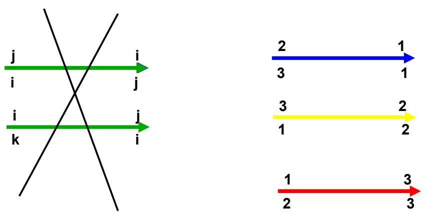
An angle-labeling is completely determined for some triangles

Outer-planar triangulation

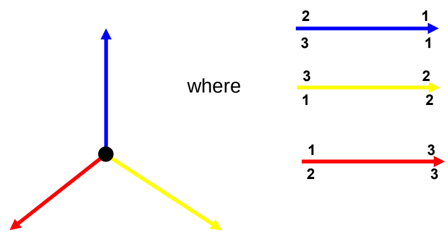


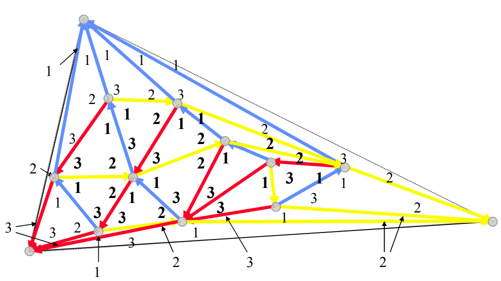
An angle-labeling is somewhat determined for other triangles

There are only 3 kinds of lines



Every interior vertex has outdegree 3





The edges of any one type (color) form a rooted tree that contains all of the interior vertices

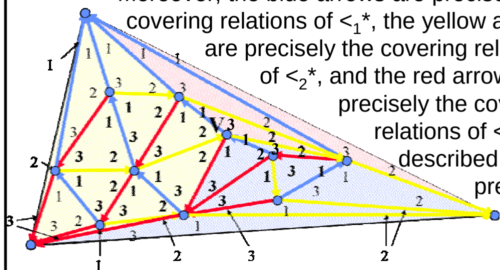
(⇒) How to convert a plane graph into 3 total orders on the vertices

Three total orders, $<_1, <_2, <_3$, with empty intersection (i.e., satisfying the vertex property) give rise to three partial orders, $<_1^*, <_2^*, <_3^*$, such that any two vertices are related to each other in exactly one of those partial orders.
$x <_i^* y \Leftrightarrow x <_i y$ & $y <_j x$ & $y <_k x$ for $\{i,j,k\} = \{1,2,3\}$

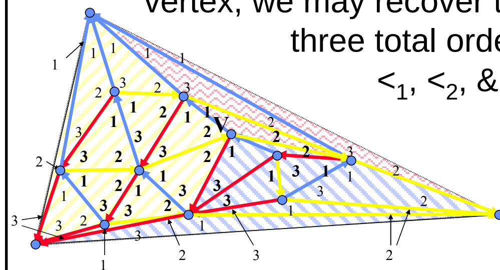
If three such partial orders each have outdegree 1, then we can recover the three total orders (see Schnyder for proof).

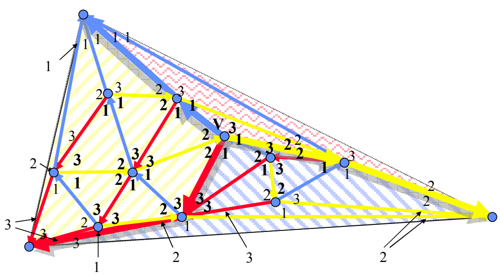
Three-tree decomposition of interior edges (Schnyder)

Moreover, the blue arrows are precisely the covering relations of $<_1^*$, the yellow arrows are precisely the covering relations of $<_2^*$, and the red arrows are precisely the covering relations of $<_3^*$, as described in the previous slide

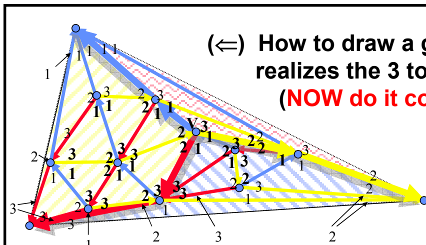


Since each of the colored arrows has out-degree 1 at each interior vertex, we may recover the three total orders $<_1$, $<_2$, & $<_3$.





The three different colored paths from V to the three extreme triangle vertices partition the triangle set into three disjoint subsets of the [2n–5] interior triangles

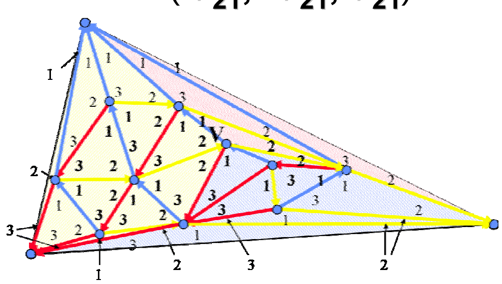


(⇐) How to draw a graph that realizes the 3 total orders (NOW do it compactly!)

Count triangles in each partition of the 3-partition of the triangulation, then divide the count by the total number of internal triangles, and interpret the value as barycentric coordinates on three affinely independent points.

Triangle-count barycentric coordinates:

$V = ({}^{7}/_{21}, {}^{11}/_{21}, {}^{3}/_{21})$



For more math (proofs), see:

Walter Schnyder, Planar Graphs and Poset Dimension, ***Order*** (5), 1989, 323-343.